We complete the study of the bifurcations of saddle/spiral bimodal linear systems, depending on the respective traces $T$ and $\tau$: one 2-codimensional bifurcation; four kinds of 1-codimensional bifurcations. We stratify the bifurcation set in the $(T, \tau)$-plane and we describe the qualitative changes of the dynamical behavior at each kind of bifurcation point.

**Keywords:** Piecewise linear system; structural stability; tangency/saddle singularities; bifurcation diagram.

1. Introduction

Piecewise linear systems constitute a class of non-linear systems which have attracted the interest of researchers because of their interesting properties and the wide range of applications from which they arise. Even the planar continuous BLDS (planar continuous bimodal linear dynamical systems, that is, two planar linear subsystems acting in complementary halfplanes, assuming continuity in the separating straight line) have complex dynamic behaviors as well as applications (see, for example, [Artes et al., 2013], [Camlibel et al., 2003], [Di Bernardo et al., 2008], [Ferrer et al., 2014] and [Llibre et al., 2013]).

Our aim is a full characterization of the planar continuous BLDS structurally stable and a systematic study of the bifurcations between them, both in terms of the coefficients of the matrices which define the system. The structural stability of a system guarantees that its qualitative behavior is preserved under small perturbations of their parameters, whereas qualitative changes occur at the bifurcation points. We point out that both concepts (structural stability and bifurcation) depend on the equivalence relation which precises the idea that two dynamical systems have the "same qualitative behavior". For example, for the equilibrium point of a single (non degenerate) planar linear system, those having positive trace and positive
determinant form a unique (structurally stable) $C^0$-class (sources) whereas they are partitioned in four $C^1$-classes (spirals and nodes as structurally stable classes; improper nodes and starred nodes as bifurcations).

Here we follow Sotomayor and Garcia [2003], where two planar continuous BLDS are equivalent if there is a homeomorphism of $\mathbb{R}^2$, preserving the separating line, which maps the orbits of a system into those of the other one and it is differentiable when restricted to finite periodic orbits (see Definition 2.3). We maintain also the nomenclature in Sotomayor and Garcia [2003].

Till now, several partial studies exist concerning equilibrium points, periodic orbits or homoclinic orbits of a planar continuous BLDS. Our aim is to complete the study of the more significative dynamical elements and to integrate all of them in a complete bifurcation diagram, in particular analyzing their persistence under small perturbations. From a theoretical point of view, up we know, there is not such a global study for a wide class of non-linear systems. Concerning applications, it supplies simple models of ordinary bifurcations and some new kinds of bifurcations, including higher codimensions.

For example, here it becomes clear that in general the periodic orbits are structurally stable and that two bifurcations are possible for disappearing: an ordinary homoclinic bifurcation and a special kind of Hopf bifurcation which we detail. Indeed, in an ordinary Hopf bifurcation the periodic orbit collapses to the equilibrium point inside it, whereas here the spiral inside the periodic orbit does not collapse but change from divergent to convergent, through a continuum of periodic orbits. In addition, we prove that beyond both bifurcations there is not a zone of structural stability, but a sequence of saddle/tangency or tangency/saddle bifurcations whose limit is the corresponding bifurcation. Moreover, this global point of view includes bifurcation of higher codimension. For example, the above bifurcation joint in a 2-codimensional bifurcation, with a continuum of periodic orbits inside an homoclinic one.

For this global study, the starting point is the reduced form of the matrices representing a continuous BLDS obtained in Ferrer et al. [2010]. Then, in Ferrer et al. [2014] we have specialized the general criteria for structural stability in Sotomayor and Garcia [2003] and we have pointed out that additional specific studies (concerning periodic orbits, saddle-loop (or homoclinic) orbits, saddle/tangency orbits and tangency/saddle orbits) are needed when one of the subsystems is a spiral.

We have focused our attention in the saddle/spiral case because it is the only one where all these elements can appear so that, more complex behaviors and applications are expected. Thus, again in Ferrer et al. [2014], we have studied the periodic orbits and the saddle-loop orbits for the case of a saddle/spiral system. For this case, in Xu et al. [2013] one specifies the conditions for the existence of saddle-loop orbits. In general, for planar continuous BLDS, in Freire et al. [1998] it is proved that there exists at most one saddle-loop orbit or limit cycle, which then must be attracting or repelling. All these previous partial results are collected in Section 3.

Here, as a first goal, in Section 4 we tackle the tangency/saddle and the saddle/tangency orbits, in order to complete the study of the dynamical elements which determine the structural stability of the systems. We conclude that they appear for a sequence of values of the trace of the spiral, converging to the ones corresponding to the above homoclinic bifurcation and degenerate Hopf bifurcation.

Finally, in Section 5, we integrate all these elements in a complete bifurcation diagram which summarizes the structural stability and the bifurcations of saddle/spiral systems, in terms of the respective traces $T$ and $\tau$. Indeed, five kinds of bifurcations are found: tangency/saddle (Theorem 4), saddle/tangency (Theorem 5), and the so-called (H), (C) and (O)-bifurcation (Theorem 2). In the (H)-bifurcation, a saddle-loop orbit appears; in the (C)-bifurcation, a continuum of periodic orbits; in the (O)-bifurcation, both kind of phenomena. For the remainder structurally stable systems, we characterize when(hyperbolic) periodic orbits exist.

Moreover, the bifurcation set in the $(T, \tau)$-plane is stratified (Theorem 6) as follows: the origin, as a 2-codimensional (O)-bifurcation; two 1-codimensional strata formed by (H)-bifurcation points; two 1-codimensional strata formed by (C)-bifurcation points; a sequence of 1-codimensional strata formed by tangency/saddle bifurcation points; a sequence of 1-codimensional strata formed by saddle/tangency bifurcation points. The origin is a limit point of the (H)-strata and the (C)-strata; in turn, they are limit
points of the sequences of tangency/saddle and saddle/tangency strata.

We include figures of each kind of bifurcation systems and of the bifurcation diagram for particular values of the parameters. We also include a structurally stable system where a (hyperbolic) periodic orbit appears.

The paper is organized as follows. In Section 2 we recall the basic definitions. In Section 3 we summarize the previous results, concerning general criteria for structural stability and the study of saddle-loop orbits and periodic orbits for saddle-spiral systems. Section 4 is devoted to the study of tangency/saddle bifurcations and saddle/tangency bifurcations. Finally, in Section 5 we analyze the different bifurcations and we describe the bifurcation diagram.

Throughout the paper, $\mathbb{R}$ will denote the set of real numbers, $M_{n \times m}(\mathbb{R})$ the set of matrices having $n$ rows and $m$ columns and entries in $\mathbb{R}$ (in the case where $n = m$, we will simply write $M_n(\mathbb{R})$) and $\text{Gl}_n(\mathbb{R})$ the group of non-singular matrices in $M_n(\mathbb{R})$. Finally, we will denote by $e_1, \ldots, e_n$ the natural basis of the Euclidean space $\mathbb{R}^n$.

2. Structural stability of planar bimodal linear systems

Let us consider a bimodal linear dynamical system given by two subsystems each one acting in a halfspace:

$$
\dot{x}(t) = A_1 x(t) + B_1 \quad \text{if } Cx(t) \leq 0,
$$

$$
\dot{x}(t) = A_2 x(t) + B_2 \quad \text{if } Cx(t) \geq 0,
$$

where $A_1, A_2 \in M_n(\mathbb{R})$; $B_1, B_2 \in M_{n \times 1}(\mathbb{R})$; $C \in M_{1 \times n}(\mathbb{R})$. We assume that the dynamics is continuous along the separating hyperplane $H = \{ x \in \mathbb{R}^n : Cx = 0 \}$; namely, that both subsystems coincide for $C x(t) = 0$.

By means of a linear change in the state variable $x(t)$, we can consider $C = (1 \ 0 \ldots \ 0) \in M_{1 \times n}(\mathbb{R})$. Hence $H = \{ x \in \mathbb{R}^n : x_1 = 0 \}$ and continuity along $H$ is equivalent to:

$$
B_2 = B_1, \quad A_2 e_i = A_1 e_i, \quad 2 \leq i \leq n.
$$

We will write from now on $B = B_1 = B_2$.

**Definition 2.1.** In the above conditions, we say that the triple of matrices $(A_1, A_2, B)$ defines a continuous *bimodal linear dynamical system* (BLDS.)

The placement of the equilibrium points will play a significative role in the dynamics of a BLDS. So, we define:

**Definition 2.2.** Let us assume that a subsystem of a BLDS has a unique equilibrium point, not lying in the separating hyperplane. We say that this equilibrium point is *real* if it is located in the halfspace corresponding to the considered subsystem. Otherwise, we say that the equilibrium point is *virtual*.

Our goal is to characterize the planar continuous BLDS which are structurally stable in the sense of Sotomayor and Garcia [2003] in terms of the coefficients $A_1, A_2$ and $B$, and to analyze the bifurcations appearing in the boundary values between them.

**Definition 2.3.** A triple of matrices $(A_1, A_2, B)$ defining a continuous BLDS is said to be (regularly) *structurally stable* if it has a neighborhood $V(A_1, A_2, B)$ such that for every triplet $(A_1', A_2', B') \in V(A_1, A_2, B)$ there is a homeomorphism of $\mathbb{R}^2$ preserving the hyperplane $H$ which maps the oriented orbits of $(A_1', A_2', B')$ into those of $(A_1, A_2, B)$ and it is differentiable when restricted to finite periodic orbits.

A natural tool in the study of BLDS is simplifying the matrices $A_1, A_2, B$ by means of changes in the variables $x(t)$ which preserve the qualitative behavior of the system (in particular, the condition of structurally stability). So, we consider linear changes in the state variables space preserving the hyperplanes...
$x_1(t) = k$, which will be called \textit{admissible basis changes}. Thus, they are basis changes given by a matrix $S \in \text{Gl}_n(\mathbb{R})$, 

$$S = \begin{pmatrix} 1 & 0 \\ U & W \end{pmatrix}, \quad W \in \text{Gl}_{n-1}(\mathbb{R}), \quad U \in \mathbb{M}_{n-1 \times 1}(\mathbb{R}).$$

See Ferrer et al. [2010] for the resulting reduced forms.

Also, translations parallel to the hyperplane $H$ are allowed.

### 3. Preliminaries

By specializing to BLDS the general necessary and sufficient conditions in Sotomayor and Garcia [2003], in Ferrer et al. [2014] one proves the following results.

**Theorem 1.** [Ferrer et al., 2014] Let us consider planar continuous BLDS.

1. If such a BLDS is structurally stable, then the triples of matrices representing it can be reduced (by means of an admissible basis change and a translation parallel to the separating line) to the form:

   $$A_1 = \begin{pmatrix} T & 1 \\ -D & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \tau & 1 \\ -\Delta & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ b \end{pmatrix}, \quad b \neq 0$$

   In particular, the only tangency point is $(0,0)$.

2. If one of the subsystems is a center, a degenerate node, an improper node or a starred node, then the BLDS is not structurally stable.

3. For the remainder BLDS, if none subsystem is a real spiral then the BLDS is structurally stable. Explicitly (for $b > 0$; when $b < 0$, we obtain the symmetric ones) when:

   - the left subsystem is a real saddle, a virtual node or a virtual spiral
   - the right subsystem is a virtual saddle or a real node

4. Additional conditions must be verified if one of the subsystems is a real spiral (in the right halfplane if $b > 0$):

   4.1 A BLDS real saddle/real spiral is structurally stable if and only if:
   - the finite periodic orbits are hyperbolic
   - there are not saddle-loop orbits
   - there are not finite orbits connecting a saddle and a tangency point

   4.2 A BLDS virtual node/real spiral is structurally stable if and only if condition (a) holds

   4.3 A BLDS virtual spiral/real spiral is structurally stable if and only if condition (a) holds and also:
   - the infinite periodic orbit at infinity is hyperbolic

**Remark 3.1.** In (1) of the above Theorem one can take $b = 1$ (by means of a change of scale and a symmetry, if necessary), but we will consider general $b \neq 0$ because of the homogeneity in the obtained formulas.

The cases (4.1), (4.2), (4.3) need additional specific studies. In Ferrer et al. [2014] one focuses on conditions (a), (b) of case (4.1) for divergent spirals ($\tau > 0$). Thus, let us assume a BLDS as in (1) of Theorem 1, verifying:

- The left subsystem is a (real) saddle, i.e.: $D < 0, b > 0$. In particular, its equilibrium point is $(\frac{b}{D}, -\frac{b}{D})$, and the invariant manifold cut the separating line at $(0, -\frac{b}{\Delta_2})$ and $(0, -\frac{b}{\Delta_1})$, where $\lambda_2 < 0 < \lambda_1$ are the eigenvalues of $A_1$ ($\lambda_1 + \lambda_2 = T$, $\lambda_1\lambda_2 = D$.)

- The right subsystem is a (real) spiral, i.e.: $\Delta > 0, \tau^2 < 4\Delta, b > 0$. In particular, its equilibrium point is $(\frac{b}{\Delta}, -\frac{\tau}{\Delta})$. We write $\alpha = \pm i\beta, \beta > 0$ the eigenvalues of $A_2$ ($2\alpha = \tau, \alpha^2 + \beta^2 = \Delta$.)

We summarize the results in Ferrer et al. [2014] concerning this case (4.1). Indeed, we precise the uniqueness of the finite periodic orbit in (2.c) (see [Freire \textit{et al.}, 1998]).
Theorem 2. [Ferrer et al., 2014] As above, let us assume:
\[ b > 0, D < 0, \Delta > 0, \tau^2 < 4\Delta \]
in (1) of Theorem 1, and let be
\[ \lambda_2 < 0 < \lambda_1 \quad \text{the eigenvalues of } A_1 \]
\[ \alpha \pm i\beta, \beta > 0, \quad \text{the eigenvalues of } A_2 \]
Then, for \( \tau > 0 \):

1. If \( T \geq 0 \), there are not homoclinic orbits nor finite periodic orbits.
2. If \( T < 0 \), the only homoclinic (i.e., saddle-loop) orbit appears for the value \( \tau_H \) of \( \tau \) verifying

\[ t = \frac{1}{\tau} \ln \left( \frac{\lambda_2^2 - \tau \lambda_1 + \Delta}{\lambda_2^2 - \tau \lambda_2 + \Delta} \right) \]

being

\[ \exp(\alpha t) \sin(\beta t - \varphi) + \frac{\beta}{M} = 0, \quad \pi + \varphi \leq \beta t \leq \frac{3\pi}{2} + \varphi \]

where \( M > 0 \) and \( 0 < \varphi < \pi \) are defined by

\[ M \cos(\varphi) = \alpha - \frac{\Delta}{\lambda_2}, \quad M \sin(\varphi) = \beta. \]

Moreover, \( \tau_H < T_D^\Delta \).

3. If \( T < 0 \), a unique finite periodic orbit exists for each \( 0 < \tau < \tau_H \), being attractive hyperbolic (and transverse to the separating axis). No saddle/tangency orbits exist.

Corollary 3.1. [Ferrer et al., 2014] The systems in case (4.1) of Theorem 1 with \( T < 0 \) and \( 0 < \tau < \tau_H \) are structurally stable.

Example 3.1. For \( T = -1, D = -1, \Delta = 5, b = 1 \), one obtains: \( \tau_H = 0.742 \) (see Fig. 4.H). For \( \tau = 0.1 \) (in general, for \( 0 < \tau < \tau_H \)), a periodic attractive orbit exists (see Fig. 4.P).

The following theorem specifies the value of \( \tau_H \).

Theorem 3. [Xu et al., 2013] In the conditions of Theorem 2, \( \tau_H \) is the value of \( \tau \) verifying

\[ \frac{1}{2} \ln \left( \frac{\lambda_2^2 - \tau \lambda_1 + \Delta}{\lambda_2^2 - \tau \lambda_2 + \Delta} \right) - \frac{\alpha}{\beta} (2\pi - \arctan \frac{\lambda_2 \alpha - \Delta}{\lambda_2 \beta} - \arctan \frac{\Delta - \lambda_1 \alpha}{\lambda_1 \beta}) = 0. \]

Remark 3.2. For \( \tau < 0 \) one has symmetric results:

1'. If \( T \leq 0 \), there are not homoclinic orbits nor finite periodic orbits.
2'. If \( T > 0 \), the only homoclinic orbit appears for a unique value \( \tau_H < 0 \) of \( \tau \) verifying \( \tau_H > T_D^\Delta \).
3'. If \( T > 0 \), a unique finite periodic orbit exists for each \( \tau_H < \tau < 0 \), being hyperbolic (and transverse to the separating axis). Hence, the system is structurally stable.

In the proof of Theorem 2 and in Section 4, the following lemma is used:

Lemma 1. [Ferrer et al., 2014] A spiral cuts \( x_1 = 0 \) in \( x_{21} \) and \( x_{22} \) if and only if, for some \( t \)

\[ \exp(\mu t) = \frac{b + \mu x_{22}}{b + \mu x_{21}} \]

where \( \mu = \alpha + i\beta, \beta > 0 \), is an eigenvalue of the spiral.
4. The tangency/saddle and the saddle/tangency singularities

In this section we study the case $\tau > \tau_H$ and $\tau > 0$, for $T < 0$ (see Remark 4.1 for $T > 0$), where tangency/saddle and saddle/tangency singularities will appear. We recall (see (1) of Theorem 1) that there is just one tangency orbit, that is an orbit which is tangent to the separating axis, being the origin the tangency point. A tangency/saddle (respectively, saddle/tangency) singularity occurs if this tangency orbit passes through the point $(0, -b/\lambda_2)$ (respectively, $(0, -b/\lambda_1)$), so that it goes to (respectively, it comes from) the saddle equilibrium point.

We will use the following lemma:

**Lemma 2.** Consider a spiral cutting $x_1 = 0$ in $x_21$ and $x_22$. As above, let $\mu = \alpha + i\beta$, $\beta > 0$, be an eigenvalue and we denote $2\alpha = \tau$, $\alpha^2 + \beta^2 = \Delta$.

(i) The time for going from $x_21$ to $x_22$ is

$$t = \frac{1}{\tau} \ln\left( \frac{(b + \alpha x_{22})^2 + (\beta x_{22})^2}{(b + \alpha x_{21})^2 + (\beta x_{21})^2} \right)$$

(ii) $x_{22}$ is the first intersection (after $x_21$) of the spiral with $x_1 = 0$ if and only if

$$\exp(\alpha t) \sin(\beta t - \varphi) + \frac{\beta}{M} = 0, \quad \pi + \varphi \leq \beta t \leq \frac{3\pi}{2} + \varphi$$

being $M > 0$ and $0 < \varphi < \pi$ defined by

$$M \cos(\varphi) = \alpha + \Delta b x_{21}, \quad M \sin(\varphi) = \beta.$$

**Proof.**

(i) From Lemma 1, by taking the square of the modulus of both sides of the equation, we have:

$$\exp(2\alpha t) = \frac{(b + \alpha x_{22})^2 + (\beta x_{22})^2}{(b + \alpha x_{21})^2 + (\beta x_{21})^2}$$

and from it we get the expression for $t$.

(ii) Again from Lemma 1, we isolate $x_{22}$ and multiply and divide by the conjugate of $\mu$

$$x_{22} = \frac{b}{\Delta} \left( \exp(\alpha t)(\cos(\beta t)(\alpha + \frac{\Delta}{b} x_{21}) + \beta \sin(\beta t)) - \alpha \right)$$

$$+ i(\exp(\alpha t)(\sin(\beta t)(\alpha + \frac{\Delta}{b} x_{21}) - \beta \cos(\beta t)) + \beta)$$

We define $M > 0$ and $0 < \varphi < \pi$ by

$$M \cos(\varphi) = \alpha + \frac{\Delta}{b} x_{21}, \quad M \sin(\varphi) = \beta,$$

so that

$$x_{22} = M \frac{b}{\Delta} \left( \exp(\alpha t)(\cos(\beta t - \varphi) - \frac{\alpha}{M}) ight)$$

$$+ i(\exp(\alpha t)(\sin(\beta t - \varphi) + \frac{\beta}{M}))$$

From it, the first intersection of the spiral with $x_1 = 0$ must verify

$$\exp(\alpha t) \sin(\beta t - \varphi) + \frac{\beta}{M} = 0$$

and

$$\pi + \varphi \leq \beta t \leq \frac{3\pi}{2} + \varphi.$$
First, we tackle the values $\tau > \tau_H$. We will see that there is a decreasing sequence $\tau_1, \tau_2, \ldots \rightarrow \tau_H$ of values of $\tau$ where tangency/saddle singularities appear. For the remainder values, the BLDS is structurally stable.

**Theorem 4.** Let us assume:

\[ b > 0, D < 0, \Delta > 0, \tau^2 < 4\Delta \]

in (1) of Theorem 1, and let be

\[ \lambda_2 < 0 < \lambda_1 \quad \text{the eigenvalues of } A_1 \]

\[ \alpha \pm i\beta, \beta > 0, \quad \text{the eigenvalues of } A_2 \]

Then, for $T < 0$ and $\tau > \tau_H (> 0)$:

1. There exists a maximal value of $\tau$, $\tau_1$ (see Fig. 1), for which a tangency/saddle orbit appears. It is the only value of $\tau$ for which the tangent orbit at $(0,0)$ has its first intersection with the separating line just at $(0, -b/\lambda_1)$. That is, the value of $\tau > 0$ verifying

\[ t = \frac{1}{\tau} \ln\left(\frac{\lambda_1^2 - \lambda_1 \tau + \Delta}{\lambda_1^2}\right) \]

being

\[ \exp(\alpha t) \sin(\beta t - \varphi) + \frac{\beta}{M} = 0, \quad \pi + \varphi \leq \beta t \leq \frac{3\pi}{2} + \varphi \]

where $M > 0$ and $0 < \varphi < \pi$ are defined by

\[ M \cos(\varphi) = \alpha, \quad M \sin(\varphi) = \beta. \]

Moreover, $\tau_1 < \frac{\Delta}{\lambda_1}$.

2. There exists a decreasing sequence $(\tau_1, \tau_2, \ldots, \tau_k, \ldots) \rightarrow \tau_H$, $k \geq 1$, for which tangency/saddle orbits appear.

For the value $\tau = \tau_k$ the orbit through the tangency point $(0,0)$ has its $(2k - 1)$th intersection with the separating line just at $(0, -b/\lambda_1)$. (see Fig. 1)

3. For the remainder values of $\tau > \tau_H$, the BLDS is structurally stable.

**Proof.**

1. Using Lemma 2, and imposing that $x_{21} = 0$ and $x_{22} = -b/\lambda_1$ we get the above expressions.

And, from $t = \frac{1}{\tau} \ln\left(\frac{\lambda_1^2 - \lambda_1 \tau + \Delta}{\lambda_1^2}\right)$, we get the bound for $\tau_1$.

2. For $\tau = \tau_1$ the tangent orbit at $(0,0)$ intersects $x_1 = 0$ just at $(0, -b/\lambda_1)$ (see Fig. 1).

Now we prove the following claim: when $\tau$ decreases, this (first) intersection point ascends.

From Lemma 1, if the orbit is starting at $(0,0)$, we have

\[ \exp(\mu t) = \frac{b + \mu x_{22}}{b} \]

or, equivalently, taking into account that $\mu = \alpha + i\beta$, and writing $y = -x_{22} > 0$:

\[ b \exp(\alpha t) \cos(\beta t) + \alpha y = b \]

\[ b \exp(\alpha t) \sin(\beta t) + \beta y = 0 \]
Moreover, $\alpha^2 + \beta^2 = \Delta$ so that $\beta$ depends implicitly on $\alpha$, and $\frac{\partial \beta}{\partial \alpha} = -\frac{\alpha}{\beta}$.

We consider $f(\alpha, y, t) = (f_1, f_2)$, being

$$
\begin{align*}
  f_1 &= b \exp(\alpha t) \cos(\beta t) + \alpha y - b \\
  f_2 &= b \exp(\alpha t) \sin(\beta t) + \beta y
\end{align*}
$$

where one assumes $\beta$ depending implicitly on $\alpha$ as above. The equations $f(\alpha, y, t) = (0, 0)$ define $y, t$ as implicit functions of $\alpha$ because:

$$
\det \frac{\partial f}{\partial (y, t)} = \det \begin{pmatrix} \alpha & \partial_{f_1} \\ \beta & \partial_{f_2} \end{pmatrix} = b\Delta \exp(\alpha t) \sin(\beta t) \neq 0
$$

being

$$
\begin{align*}
  \partial_{f_1} &= b\alpha \exp(\alpha t) \cos(\beta t) - b\beta \exp(\alpha t) \sin(\beta t) \\
  \partial_{f_2} &= b\alpha \exp(\alpha t) \sin(\beta t) + b\beta \exp(\alpha t) \cos(\beta t)
\end{align*}
$$

We compute next

$$
\frac{\partial f}{\partial \alpha} = \begin{pmatrix} \partial_{f_1} \\ \partial_{f_2} \end{pmatrix}
$$

being

$$
\begin{align*}
  \partial_{f_1} &= b\alpha \exp(\alpha t) \cos(\beta t) + bt \exp(\alpha t) \sin(\beta t) \frac{\beta}{\gamma} + y \\
  \partial_{f_2} &= b\alpha \exp(\alpha t) \sin(\beta t) - bt \exp(\alpha t) \cos(\beta t) \frac{\beta}{\gamma} - \frac{\gamma}{\beta} y
\end{align*}
$$

and

$$
\left( \frac{\partial f}{\partial (y, t)} \right)^{-1} = -\frac{1}{b\Delta \exp(\alpha t) \sin(\beta t)} \begin{pmatrix} G_1 & G_2 \\ -\beta & \alpha \end{pmatrix}
$$

being

$$
\begin{align*}
  G_1 &= b\alpha \exp(\alpha t) \sin(\beta t) + b\beta \exp(\alpha t) \cos(\beta t) \\
  G_2 &= b\beta \exp(\alpha t) \sin(\beta t) - b\alpha \exp(\alpha t) \cos(\beta t)
\end{align*}
$$

Since

$$
\frac{\partial (y, t)}{\partial \alpha} = -\left( \frac{\partial f}{\partial (y, t)} \right)^{-1} \frac{\partial f}{\partial \alpha}
$$

one has

$$
\frac{\partial y}{\partial \alpha} = -\frac{b \exp(\alpha t) 2\beta t - \sin(2\beta t)}{2\beta^2 \sin(\beta t)} < 0
$$

for all $t > 0$, as we claimed (we recall $\tau = 2\alpha$, $y = -x_{22}$).

When $\tau$ decreases under $\tau_1$, this (first) intersection point ascends over $(0, -b/\lambda_1)$, so that the orbit completes a full turn and intersects the axis $x_1 = 0$ twice between $(0, -b/\lambda_2)$ and $(0, -b/\lambda_1)$, and again under $(0, -b/\lambda_1)$ if $\tau - \tau_1$ is small enough. Analogously to the above claim this third intersection point is just $(0, -b/\lambda_1)$ (see Fig. 1).

Additional decrease of $\tau$ gives a second turn (with two additional intersections between $(0, -b/\lambda_2)$ and $(0, -b/\lambda_1)$) and a fifth intersection with $x_1 = 0$ under $(0, -b/\lambda_1)$. As above, for a certain (unique) value $\tau = \tau_2$ this third intersection point is just $(0, -b/\lambda_1)$ (see Fig. 1).

By recurrence, one obtains a sequence of decreasing values $\tau_1, \tau_2, \ldots, \tau_k, \ldots$ for which the tangent orbit at $(0, 0)$ intersects $x_1 = 0$ in $0, 2, \ldots, 2k - 2, \ldots$ points between $(0, -b/\lambda_2)$ and $(0, -b/\lambda_1)$, and another one just at $(0, -b/\lambda_1)$.

An analogous reasoning shows that $\lim \tau_k = \tau_H$ for $\tau = \tau_H$ the saddle orbit through $(0, -b/\lambda_2)$ intersects again $x_1 = 0$ at $(0, -b/\lambda_1)$, whereas the tangent orbit at $(0, 0)$ turns over the spiral toward this homoclinic orbit: for any slightly greater value $\tau_H + \epsilon$ the above saddle orbit intersects $x_1 = 0$ under $(0, -b/\lambda_1)$, so that the tangent orbit passes between this new intersection point and $(0, -b/\lambda_1)$; therefore, the reasoning in the above paragraph shows that there is some $\tau_k < \tau_H + \epsilon$. 


3. By construction, for $\tau_k > \tau > \tau_{k+1}$ there are not tangency/saddle orbits. Moreover, the orbits for $\tau$ run between the ones for $\tau_k$ and $\tau_{k+1}$ so that neither finite periodic orbits nor saddle/tangency orbits can occur.

Example 4.1. For $T = -1, D = -1, \Delta = 5, b = 1$ as in Example 3.1, we plot (Fig. 1, from right to left, respectively) the tangency/saddle orbits: $\tau_1 = 1.145, \tau_2 = 0.782, \tau_3 = 0.745$. We recall $\tau_H = 0.742$.

![Fig. 1. From right to left, the tangency/saddle orbits corresponding to values: $\tau_1 = 1.145, \tau_2 = 0.782, \tau_3 = 0.745$](image)

In an analogous way, for $\tau < 0$ there is an increasing sequence $\tau_{-1}, \tau_{-2}, ... \to 0$ of values of $\tau$ where saddle/tangency singularities appear.

**Theorem 5.** Let us assume:

\[ b > 0, D < 0, \Delta > 0, \tau^2 < 4\Delta \]

in (1) of Theorem 1, and let be

\[ \lambda_2 < 0 < \lambda_1 \quad \text{the eigenvalues of } A_1 \]

\[ \alpha \pm i\beta, \beta > 0, \quad \text{the eigenvalues of } A_2 \]

Then, for $T < 0$ and $\tau < 0$:

1. There exists a minimal value of $\tau$, $\tau_{-1}$, for which a saddle/tangency orbit appears. It is the only value of $\tau$ for which the orbit through $(0, -b/\lambda_2)$ (that is, arising at the saddle) has the first intersection with the separating line (tangentially) at $(0, 0)$. That is, the value of $\tau < 0$ verifying

\[ t = \frac{1}{\tau} \ln(\frac{\lambda_2^2}{\lambda_2^2 - \lambda_2 \tau + \Delta}) \]

being

\[ \exp(\alpha t) \sin(\beta t - \varphi) + \frac{\beta}{M} = 0, \quad \pi + \varphi \leq \beta t \leq \frac{3\pi}{2} + \varphi \]

where $M > 0$ and $0 < \varphi < \pi$ are defined by

\[ M \cos(\varphi) = \alpha - \frac{\Delta}{\lambda_2}, \quad M \sin(\varphi) = \beta. \]

Moreover, $\tau_{-1} > \frac{\Delta}{\lambda_2}$.

2. There exists an increasing sequence $(\tau_{-1}, \tau_{-2}, ..., \tau_{-k}, ...) \to 0, k \geq 1$, for which saddle/tangency orbits appear.

For the value $\tau = \tau_{-k}$ the orbit through $(0, -b/\lambda_2)$ has its $(2k - 1)^{th}$ intersection with the separating line just at the origin.
3. For the remainder values of $\tau < 0$, the BLDS is structurally stable.

Proof. Analogously than in the proof of Theorem 4. ■

Remark 4.1. For $T > 0$ one has symmetric results:

(1') For $\tau > \tau_H$:
- There exists a minimal value of $\tau$, $\tau_1$, for which a tangency/saddle orbit appears, verifying $\tau_1 > \frac{\Delta_2}{2}$.
- There exists an increasing sequence $(\tau_1, \tau_2, ..., \tau_k, ...)$ $\to \tau_H$, $k \geq 1$, for which tangency/saddle orbits appear.
- For the remainder values of $\tau < \tau_H$, the BLDS is structurally stable.

(2') For $\tau < 0$:
- There exists a maximal value of $\tau$, $\tau_1$, for which a saddle/tangency orbit appears, verifying $\tau_1 < \frac{\Delta_1}{2}$.
- There exists a decreasing sequence $(\tau_1, \tau_2, ..., \tau_k, ...)$ $\to 0$, $k \geq 1$, for which saddle/tangency orbits appear.
- For the remainder values of $\tau > \tau_H$, the BLDS is structurally stable.

5. Bifurcation diagram

For single spirals, the trace $\tau$ is a $C^0$-bifurcation parameter, being $\tau = 0$ the unique bifurcation value: for $\tau \neq 0$ the equilibrium point is hyperbolic (hence, the spirals are structurally stable), whereas it is not for $\tau = 0$ (the equilibrium point is a center); the transition transforms the character stable/unstable of the equilibrium point, towards a center.

Combining Theorems 2, 4 and 5, (where one assumes $T < 0$) we see that for saddle/spiral systems the trace $\tau$ of the spiral is also a bifurcation parameter. But the bifurcation set is more complex:
- the value $\tau = 0$ (a center, as in the single case)
- the value $\tau = \tau_H > 0$ (a saddle-loop orbit)
- a decreasing sequence $(\tau_1, \tau_2, ...)$ $\to \tau_H$
- an increasing sequence $(\tau_{-1}, \tau_{-2}, ...)$ $\to 0$

For the remainder values of $\tau$, the system is structurally stable, having periodic (hyperbolic) orbits only for $0 < \tau < \tau_H$. Notice that $\tau = 0$ and $\tau = \tau_H$ are accumulation points of the bifurcation set, so that none neighborhood is entirely formed by structurally stable systems.

Let us analyze the dynamic behavior at the bifurcation points. Bearing in mind Theorem 2 and Remark 3.2, we define:

Definition 5.1. In the conditions of Theorem 2, we will call

(1) (H)-bifurcation the one appearing for $\tau = \tau_H$. ($T \neq 0$)
(2) (C)-bifurcation the one appearing for $\tau = 0$. ($T \neq 0$)
(3) (O)-bifurcation the one appearing for $\tau = T = 0$.
(4) (TS)-bifurcations the tangency/saddle in the decreasing sequence $(\tau_1, \tau_2, ...)$ $\to \tau_H$.
(5) (ST)-bifurcations the saddle/tangency in the increasing sequence $(\tau_{-1}, \tau_{-2}, ...)$ $\to 0$.

The most interesting changes appear in one of the four quadrants determined by the invariant lines of the saddle: the one containing the spiral equilibrium point which we will call the main quadrant. For example, the (hyperbolic) periodic orbits for $0 < \tau < \tau_H$ are placed in it (see Fig. 4.P). We will denote by $V$ the vertical segment

$$V = [(0, -b/\lambda_1), (0, -b/\lambda_2)]$$

intersection of the separating axis with the open main quadrant.

5.1. The tangency/saddle and the saddle/tangency bifurcations ($T < 0$)

We have seen (Theorem 4) that for $\tau = \tau_H$ one has a tangency/saddle bifurcation. Indeed, the orbit tangent to the separating line at the origin intersects $V$ transversality $2k - 2$ times. The following intersection with the separating line is just at $(0, -b/\lambda_1)$, going then towards the saddle equilibrium point (see Fig. 1).
Analogously, for $\tau = \tau_{-k}$ one has a saddle/tangency bifurcation: the orbit through $(0, -b/\lambda_2)$, coming from the saddle equilibrium point, intersects transversality $2k - 2$ times the vertical segment $V$; the following intersection is tangent at the origin; afterwards, it spirales (in the spiral half-plane) towards the spiral equilibrium point (see Fig. 4.ST$_1$).

We have pointed out that for $\tau = \tau_k$ the tangency orbit, after passing through the origin, crosses $2k - 2$ times the segment $V$, being the following intersection with the separating axis just at the (low) frontier point of $V$, $(0, -b/\lambda_1)$. We add that the BLDS having $\tau_{k+1} < \tau < \tau_k$ form an equivalence class of structurally stable BLDS, characterized by the fact that the tangency orbit crosses $2k$ times the segment $V$ (after passing through the origin), but none its frontier. Analogously for $\tau_{-k} < \tau < \tau_{-(k+1)}$, where the tangency orbit crosses the separating axis before passing through the origin. Notice that for $0 < \tau < \tau_H$, the tangency orbit crosses infinitely many times the segment $V$.

### 5.2. The (H)-bifurcation at $\tau = \tau_H$ ($T < 0$)

See Fig. 2. We have called the (H)-bifurcation the one at $\tau = \tau_H$. It is similar to the classical homoclinic one in the following sense: when $\tau < \tau_H$ increases, the periodic orbit (surrounding the spiral equilibrium point and crossing the vertical segment $V$) increase, become a saddle-loop orbit at $\tau = \tau_H$, and disappear (as periodic orbit) for $\tau > \tau_H$; the (unstable) spiral equilibrium point rests unchanged.

On the other hand, for decreasing values of $\tau > \tau_H$, the tangency orbit spirales more and more into the main quadrant (see 5.1), before coming out of it and going to infinity (except for $\tau = \tau_k$, when the tangency orbit goes to the saddle equilibrium point). For $\tau = \tau_H$, the tangency orbit becomes bounded, spiraling towards the saddle-loop orbit; for $\tau < \tau_H$, it spirales towards the periodic orbit.

**Example 5.1.** For $T = -1$, $D = -1$, $\Delta = 5$, $b = 1$ as in the previous examples, we plot (Fig. 2, from left to right) the (H)-transition: $\tau = 0.1$, $\tau = \tau_H = 0.742$, $\tau = 0.85$.

![Fig. 2. From left to right, the (H)-transition corresponding to values: $\tau = 0.1$, $\tau = \tau_H = 0.742$, $\tau = 0.85$](image)

### 5.3. The (C)-bifurcation at $\tau = 0$ ($T < 0$)

See Fig. 3. We have called the (C)-bifurcation the one at $\tau = 0$. It is in some sense a degenerate Hopf bifurcation: for decreasing $\tau > 0$ one has the referred periodic orbit, surrounding the unstable spiral equilibrium point; for $\tau < 0$, there are not periodic orbits, and the spiral equilibrium point becomes stable.

However, the periodic orbit does not collapse, but becomes tangent to the separating line at the origin. The divergent spirals inside them for $\tau > 0$ are transformed at $\tau = 0$ in a continuum of periodic orbits (all of them placed in the spiral half-plane) and in convergent spirals for $\tau < 0$, which joint with the external convergent spiral.

Again, for increasing values $\tau < 0$, the escaping saddle orbit spirales more and more into the main quadrant towards the spiral equilibrium point and at $\tau = 0$ spirales towards the created periodic tangent orbit.

**Example 5.2.** For $T = -1$, $D = -1$, $\Delta = 5$, $b = 1$ as in the previous examples, we plot (Fig. 3, from left to right) the (C)-transition: $\tau = -0.2$, $\tau = \tau_H = 0$, $\tau = 0.172$. 
5.4. The \((O)\)-bifurcation at \(\tau = T = 0\)

In addition, the trace \(T\) of the saddle is also a bifurcation parameter (it is not for single saddle). Indeed, we have described the bifurcation set for \(T < 0\). For \(T > 0\), as we have remarked, the dynamical behavior is symmetric, being \(\tau_H < 0\). For \(T = 0\), one has \(\tau_H = 0\), so that both the decreasing sequence of the tangency/saddle bifurcations and the increasing one of the saddle/tangency bifurcations converge to \(\tau = \tau_H = 0\). Thus, for \(T = 0\) and \(\tau = 0\) one has a 2-codimensional bifurcation which we have called a \((O)\)-bifurcation: a saddle-loop, being periodic all the orbits inside it (see Fig. 4.O).

Thus, different 1-bifurcation behaviors occur, depending on the way to attempt \(T = \tau = 0\). For example, if \(T = 0\), both for \(\tau < 0\) and \(\tau > 0\), the orbits in the main quadrant spiral more and more (similarly to 5.1 and Fig. 1), becoming all of them periodic orbits for \(\tau = 0\).

If \(\tau = 0\), \(T \neq 0\), one has the referred \((C)\)-bifurcation (see Fig. 4.C): a continuum of periodic orbits inside the one tangent to the separating line at the origin; the remaining orbits in the main quadrant spiral surrounding the tangent orbit. When \(T = 0\), all these spirals become also periodic orbits, inside the (new) saddle-loop.

If \(\tau = \tau_H(T), T \neq 0\), one has the referred \((H)\)-bifurcation (see Fig. 4.H): a saddle-loop, being non-periodic any of the spirals inside it. When \(T = 0\), all these spirals become periodic orbits.

Finally, if \(0 < \tau < \tau_H(T), T \neq 0\), one has a structurally stable system (see Fig. 4.P): there are not saddle-loop orbits, neither a continuum of periodic orbits, but a hyperbolic periodic orbit (in the main quadrant). When \(T = 0\), this periodic orbit become a saddle-loop orbit, and all the spirals inside it become periodic orbits.

5.5. Bifurcation diagram

Summarizing, we have the following theorem (see Fig. 5):

**Theorem 6.** Let us consider a saddle/spiral BLD as the case (4.1) in Theorem 1, and let \(T, \tau\) be the respective traces. Then the bifurcation set in the plane \((\tau, T)\) is stratified as follows:

1. The origin: a 2-codimensional \((O)\)-bifurcation.
2. The 1-codimensional \((C)\)-bifurcations form the half-axes:
   \[
   C^+ = \{(0, T), T > 0\}, \quad C^- = \{(0, T), T < 0\}.
   \]
3. The 1-codimensional \((H)\)-bifurcations form the curves:
   \[
   H^+ = \{\tau_H(T), T > 0\}, \quad H^- = \{\tau_H(T), T < 0\}.
   \]
4. The 1-codimensional tangency/saddle bifurcations form a sequence of curves:
   \[
   TS_i = \{\tau_i(T), T\}, \quad i = 1, 2, \ldots
   \]

They converge to the above ones

\[
TS_i \to C^+ \cup \{(0, 0)\} \cup H^-
\]

in the sense that:

\[
(\tau_i(T), T) \to (0, T) \text{ if } T > 0; \quad (\tau_i(0), 0) \to (0, 0); \quad (\tau_i(T), T) \to (\tau_H(T), T) \text{ if } T < 0.
\]
(5) The 1-codimensional saddle/tangency bifurcations form a sequence of curves:

\[ ST_i = \{ (\tau_i(T), T) \} \]

They converge to the above ones

\[ ST_i \to H^+ \cup \{(0,0)\} \cup C^- \]

in the sense that:

\[ (\tau_{-1}(T), T) \to (\tau_H(T), T) \text{ if } T > 0; \quad (\tau_{-1}(0), 0) \to (0,0); \quad (\tau_{-1}(T), T) \to (0,T) \text{ if } T < 0. \]

Proof. (1),(2) and (3) follows from Theorem 2 and Remark 3.2: for each \( T \neq 0 \), we have a unique (H)-bifurcation, for \( \tau = \tau_H \), and a unique (C)-bifurcation, for \( \tau = 0 \). The (O)-bifurcation appears when \( T = 0 \).

(4) follows from Theorem 4 (a decreasing sequence \((\tau_i) \to \tau_H\), for each \( T < 0 \)) and (2') of the Remark 4.1 (a decreasing sequence \((\tau_i) \to 0\), for each \( T > 0 \)). We recall that \( \tau_H = 0 \) for \( T = 0 \).

Analogously, (5) follows from Theorem 5 and (1') of the Remark 4.1.

Remark 5.1. Notice that the structurally stable systems constitute an open and dense set in the \((T, \tau)\)-plane, although the bifurcation set is not finitely stratified. Moreover, any neighborhood of the 2-codimensional bifurcation point (at the origin) contains the four kinds of 1-codimensional bifurcation points.

Example 5.3.

For \( D = -1, \Delta = 5, b = 1 \) as in the previous examples, we plot:

- For \( T = -1 \) (see Fig. 4), different kinds of bifurcation points: (C) for \( \tau = 0 \); (H) for \( \tau = 0.742 \); \((TS_1)\) for \( \tau = 1.145 \); \((ST_1)\) for \( \tau = -0.409 \). We plot also (see (P)) a structurally stable system, having a periodic (attractive) orbit for \( \tau = 0.1 \).
- For \( T = 0 \) (see Fig. 4): the (O)-bifurcation for \( \tau = 0 \).
- The bifurcation set in the \((\tau, T)\)-plane (see Fig. 5). Bifurcations are represented as continuous lines, while the rest of the points of the plane, including the auxiliary dot-dashed lines, represent structurally stable systems.
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References
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Fig. 5. Bifurcation diagram corresponding to Example 5.3